SOME THREE-DIMENSIONAL GAS FLOWS ADJACENT TO REGIONS OF REST

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We study some three-dimensional gas flows near the characteristic surface extended over a homogeneous polytropic gas at rest.

Flows originating near the characteristic surface of arbitrary form extended over the gas at rest, were studied for the plane problems in [1], including the case when the interface of the perturbed region and the region at rest, was found to be a surface of weak discontinuity of the basic gasdynamic quantities.

Below we consider, in addition to the case of a weak discontinuity, the case when the characteristic surface is a surface of strong discontinuity. This corresponds to the propagation of normal detonation waves (the Chapman – Jouguet condition holds at the wafe front).

Solutions are constructed using the class of three-dimensional potential double waves whose equations were first obtained in [2]. Using the double waves we can only deal with the case when the characteristic surface will, at any instant t, be a developable surface (S) in the physical (x_1, x_2, x_3)-space (obviously the plane case is included completely, without any restrictions imposed on the form of the surface).

Generally speaking, we can construct, for a given characteristic surface, an infinite set of flows in its vicinity. We settle the problem of inclusion of the double wave flows into the class of arbitrary, sufficiently smooth flows corresponding to a given characteristic surface. For this purpose we derive and solve the transfer equation for the discontinuities of normal derivatives of the basic functions, which holds along any characteristic lying on the characteristic surface.

We also show, that for sufficiently long periods of time, the flow near an arbitrary characteristic surface (S) can approximately be regarded as a double wave.

This is valid for both, the surface of weak discontinuity (S) and for the flow behind a normal detonation wave.

1. Equations of three-dimensional double waves can be written in the hodograph plane of velocities u_1 and u_2 as [2 and 3]:

$$R_{11}\Psi_{22} - 2R_{12}\Psi_{12} + R_{22}\Psi_{11} = 0 \tag{1.1}$$

$$R_{11}(\Gamma_{22} + 1 + \Psi_2^2) - 2R_{12}(\Gamma_{12} + \Psi_1\Psi_2) + R_{22}(\Gamma_{11} + 1 + \Psi_1^2) = 0$$

$$R_{11}X_{22} - 2R_{12}X_{12} + R_{22}X_{11} = 0$$

$$u_3 = \Psi(u_1, u_2), \ \Gamma(u_1, u_2) = \varkappa c^2, \quad \varkappa = 1/(\gamma - 1)$$

where γ is the specific heat ratio, c is the velocity of sound, $X(u_1, u_2)$ is the distribution function and

$$R_{ik} = -\Gamma_i\Gamma_k + \Gamma/\varkappa\,(\delta_{ik} + \Psi_i\Psi_k) \qquad (i, k = 1, 2) \tag{1.2}$$

the subscripts accompanying Ψ, Γ and X denote differentiation with respect to u_1 and u_2, and δ_{ik} is the Kronecker delta.

If the function Ψ, Γ and X are known, a flow in the physical (x_1, x_2, x_3)-space can be found from

$$(\Gamma_i + u_i + \Psi\Psi_i)\, t + X_i = x_i + \Psi_i x_3 \qquad (i=1,2) \qquad (1.3)$$

Introducing polar coordinates r and φ $(u_1 = r\cos\varphi,\ u_2 = r\sin\varphi)$, we can write (1.1) in the form

$$\begin{gathered}
\Psi_{rr}\left(-\frac{\Gamma_\varphi^2}{r^2} + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\frac{\Psi_\varphi^2}{r^2}\right) + \frac{2}{r^2}\left(\Psi_{r\varphi} - \frac{\Psi_\varphi}{r}\right)\left(\Gamma_r\Gamma_\varphi - \frac{\Gamma}{\varkappa}\Psi_r\Psi_\varphi\right) + \\
+ \frac{1}{r^2}(\Psi_{\varphi\varphi} + r\Psi_r)\left(-\Gamma_r^2 + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\Psi_r^2\right) = 0 \\
\Gamma_{rr}\left(-\frac{\Gamma_\varphi^2}{r^2} + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\frac{\Psi_\varphi^2}{r^2}\right) + \frac{2}{r^2}\left(\Gamma_{r\varphi} - \frac{\Gamma_\varphi}{r}\right)\left(\Gamma_r\Gamma_\varphi - \frac{\Gamma}{\varkappa}\Psi_r\Psi_\varphi\right) + \\
+ \frac{1}{r^2}(\Gamma_{\varphi\varphi} + r\Gamma_r)\left(-\Gamma_r^2 + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\Psi_r^2\right) - \Gamma_r^2 - \frac{\Gamma_\varphi^2}{r^2} + \\
+ 2\frac{\Gamma}{\varkappa}\left(1 + \Psi_r^2 + \frac{\Psi_\varphi^2}{r^2}\right) - \frac{1}{r^2}(\Gamma_r\Psi_\varphi - \Gamma_\varphi\Psi_r)^2 = 0 \qquad (1.4) \\
X_{rr}\left(-\frac{\Gamma_\varphi^2}{r^2} + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\frac{\Psi_\varphi^2}{r^2}\right) + \frac{2}{r^2}\left(X_{r\varphi} - \frac{X_\varphi}{r}\right)\left(\Gamma_r\Gamma_\varphi - \frac{\Gamma}{\varkappa}\Psi_r\Psi_\varphi\right) + \\
+ \frac{1}{r^2}(X_{\varphi\varphi} + rX_r)\left(-\Gamma_r^2 + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\Psi_r^2\right) = 0
\end{gathered}$$

which is more suitable for investigating the behavior of the flows defined by (1.1), at the boundary of the region of rest ($u_i = 0$, $i = 1, 2, 3$).

Let us take the velocity of sound in the unperturbed gas as unit velocity. In [4], three-dimensional double waves were used to construct flows behind the normal detonation and shock waves of constant intensity. Boundary value problems were stated and distinct solutions studied. When investigating the conditions on the line $r = 0$ for the system (1.4) which arise in the problem on the adjacency of the double wave to the region of rest we shall use, in addition to the results of [4], the following theorem.

Theorem. Instantaneous stream lines of a perturbed flow are, at any instant, orthogonal to the weak discontinuity surface over the gas at rest.

This theorem was proved for the plane case in [1] and it can be easily extended to the three-dimensional case using the kinematic conditions of compatibility on a weak discontinuity. Using this theorem and performing on r and φ the operations analogous to those performed in [1 and 4] when investigating the flows behind a shock wave, we find that the conditions

$$\begin{array}{lll}
\Psi = 0, & \Psi_\varphi = 0, & \Psi_r = \mu(\varphi) \\
\Gamma = \varkappa, & \Gamma_\varphi = 0, & \Gamma_r = \nu(\varphi) \\
X = 0, & X_\varphi = 0, & X_r = \lambda(\varphi)
\end{array} \qquad (1.5)$$

$$\nu^2(\varphi) = 1 + \mu^2(\varphi) \qquad (1.6)$$

should hold on the line $r = 0$.

Condition (1.6) is obtained on passing to the limit as $r \to 0$ in

$$\frac{|u_1(\Gamma_1 + u_1 + \Psi\Psi_1) + u_2(\Gamma_2 + u_2 + \Psi\Psi_2)|}{\sqrt{u_1^2 + u_2^2 + \Psi^2}} = 1$$

which was obtained in [4] and which corresponds to the fact that the normal velocity of propagation of a weak discontinuity is constant. Equations of motion of a weak discontinuity can be obtained from (1.3) by putting $r = 0$, and are

$$x_1 = \lambda \cos\varphi - \lambda' \sin\varphi - (\mu\cos\varphi - \mu'\sin\varphi)\, x_3 + (\nu\cos\varphi - \nu'\sin\varphi) t$$
$$x_2 = \lambda \sin\varphi + \lambda' \cos\varphi - (\mu\sin\varphi + \mu'\cos\varphi)\, x_3 + (\nu\sin\varphi + \nu'\cos\varphi)\, t \tag{1.7}$$

Combining the conditions of the developability of a ruled surface given by (1.7) in the (x_1, x_2, x_3)-space we have, at any instant $t = t_0$,

$$\begin{vmatrix} -\mu\cos\varphi + \mu'\sin\varphi & -(\mu\sin\varphi + \mu'\cos\varphi) & 1 \\ \sin\varphi\,(\mu'' + \mu) & -\cos\varphi\,(\mu'' + \mu) & 0 \\ -\sin\varphi\,[\lambda'' + \lambda + (\nu'' + \nu)\, t_0] & \cos\varphi\,[\lambda'' + \lambda + (\nu'' + \nu)\, t_0] & 0 \end{vmatrix} = 0$$

Thus we see that the surface of a weak discontinuity behind which the flow is a double wave, can only be a developable surface. The functions $\lambda(\varphi)$ and $\mu(\varphi)$ are arbitrary and can be used to specify, at any instant, an arbitrary developable surface as the surface of weak discontinuity. The values $\mu = 0$ and $\nu = \pm 1$ correspond to the plane case studied in [1].

Continuing our investigation of the problems posed for the system (1.4) with the initial conditions (1.5) on the line $r = 0$, we shall assume that the functions Ψ and Γ have continuous fourth order mixed derivatives containing second order derivatives with respect to r and φ in any order. This assumption is essential, and this property of Ψ and Γ is manifest in a number of real flows, e.g. (see also [1]) in a self-similar flow occurring behind a conical normal detonation wave generated by a point source moving with constant velocity. This flow was first investigated in [5] where it was found that a self-similar double wave adjoins, through a weak discontinuity, the region of a homogeneous gas moving with constant velocity.

Note. Obviously, all the results formulated for the flows adjacent to the region of rest through a weak discontinuity remain valid when the region of rest is replaced by the region of uniform motion.

Using the previous assumptions we can write Ψ and Γ as

$$\Psi = r\mu + {}^1/_2\, r^2 \Psi_{rr}(r_\psi, \varphi) \qquad (0 \leqslant r_\psi \leqslant r)$$
$$\Gamma = \varkappa + r\nu + {}^1/_2\, r^2 \Gamma_{rr}(r_\Gamma, \varphi) \qquad (0 \leqslant r_\Gamma \leqslant r) \tag{1.8}$$

and obtain similar expressions for Ψ_r, Ψ_φ, Γ_r, and Γ_φ. These, together with (1.6), yield the following expressions for the coefficients of (1.4)

$$-\frac{\Gamma_\varphi^2}{r^2} + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\frac{\Psi_\varphi^2}{r^2} = -\nu'^2 + 1 + \mu'^2 + O(r)$$
$$\Gamma_r \Gamma_\varphi - \frac{\Gamma}{\varkappa}\Psi_r \Psi_\varphi = O(r^2) \tag{1.9}$$
$$-\Gamma_r^2 + \frac{\Gamma}{\varkappa} + \frac{\Gamma}{\varkappa}\Psi_r^2 = r\left(-2\nu\Gamma_{rr}(r_\Gamma^*, \varphi) + 2\mu\Psi_{rr}(r_\psi^*, \varphi) + \frac{\nu^3}{\varkappa}\right) + O(r^2)$$
$$(0 \leqslant r_\Gamma^* \leqslant r,\ 0 \leqslant r_\psi^* \leqslant r)$$
$$\Gamma_r \Psi_\varphi - \Gamma_\varphi \Psi_r = r(\nu\mu' - \mu\nu') + O(r^2)$$

Multiplying all Eqs. of (1.4) by r and using (1.9) we easily see, that all coefficients of the second order derivatives in the new system are continuous at $r = 0$, and the coefficients of the second order derivatives with respect to φ are of the order $O(1)$. The line $r = 0$ will represent, for this system, a line of parabolic degeneracy and it will, in addition, be a characteristic.

The system (1.4) is nonlinear and the theorems of existence and uniqueness of solution of the problem when the initial conditions are given on the characteristic line of parabolicity, are only known for some linear systems in both, hyperbolic and elliptic cases. Here we shall attempt to obtain, within the assumptions made, approximate representations of the functions Ψ and Γ, a simplified equation for X and to investigate the problems with for given initial conditions for this equation. Solving the obtained equation we can obtain an approximate expression for X. Moreover, this equation can be used as a model in solving the problems for (1.4). In the hyperbolic case it can be shown to possess a solution. Both, the system (1.4) near $r = 0$ and the equation for X, are of the same form since the coefficients accompanying the second derivatives are the same in all Eqs. of (1.4).

Let us find approximate expressions for Ψ and Γ. Using the relations (1.9) following from the continuity of $\Psi_{rr\varphi\varphi}$, and $\Gamma_{rr\varphi\varphi}$, and the estimates

$$\Psi_{r\varphi}-\frac{\Psi_\varphi}{r}=O(r),\quad \Gamma_{r\varphi}-\frac{\Gamma_\varphi}{r}=O(r),\quad \frac{\Psi_{\varphi\varphi}}{r}=\mu''=O(r),\quad \frac{\Gamma_{\varphi\varphi}}{r}=\nu''+O(r)$$

obtained from the first two Eqs. of (1.4) by retaining in them the terms of the order $O(1)$ we obtain, for $\Psi_{rr}(0,\varphi)$ and $\Gamma_{rr}(0,\varphi)$, the following system of equations

$$\Psi_{rr}(0,\varphi)(-\nu'^2+\mu'^2+1)+(\mu''+\mu)(-2\nu\Gamma_{rr}(0,\varphi)+2\mu\Psi_{rr}(0,\varphi)+\nu^3/\varkappa)=0$$

$$\Gamma_{rr}(0,\varphi)(-\nu'^2+\mu'^2+1)+(\nu''+\nu)(-2\nu\Gamma_{rr}(0,\varphi)+2\mu\Psi_{rr}(0,\varphi)+\nu^3/\varkappa)-$$
$$-\nu^2-\nu'^2+2(\nu^2+\mu'^2)-(\nu\mu'-\mu\nu')^2=0 \qquad (1.10)$$

which yield the following approximate expressions for Ψ and Γ at small r:

$$\Psi\approx r\mu+\frac{\gamma+1}{2}\frac{\nu^3\mu^2}{\nu'^2+\mu^2}(\mu''+\mu)r^2,\quad \Gamma\approx\dot{\varkappa}+r\nu+\frac{1}{2}\left[\frac{(\gamma+1)\nu^3\mu^2(\nu''+\nu)}{\nu'^2+\mu^2}-\nu^2\right]r^2 \qquad (1.11)$$

(we note that the third order derivatives with respect to r, if they exist, cannot, in general, be determined uniquely [1]).

Assuming that X is twice continuously differentiable in r and φ at small r and using (1.9) and (1.5) together with the expressions for $\Psi_{rr}(0,\varphi)$ and $\Gamma_{rr}(0,\varphi)$, we obtain the following approximate expression for X:

$$r\chi_{rr}-\frac{(\gamma+1)\nu^3\mu^2}{\nu'^2+\mu^2}(\chi_{\varphi\varphi}+r\chi_r)=0 \qquad (1.12)$$

The sign of $\nu(\varphi)$ determines the type of this equation. Thus, if the density in the perturbed flow increases with increasing distance from the weak discontinuity Eq. (1.12) is hyperbolic for $r > 0$ (this occurs e.g. near a weak shock expanding behind a normal detonation wave). If, on the other hand, the density decreases (as it happens in a rarefaction wave), then (1.12) is elliptic for $r > 0$. In the plane case ($\mu = 0, \nu = \pm 1$) (1.12) becomes

$$r\,X_{rr}\pm(\gamma+1)(X_{\varphi\varphi}+r\,X_r)=0 \qquad (1.13)$$

In the earlier work [1] we have investigated problems for (1.13) with initial conditions (1.5), using the theorems given in [6 and 7]. We can apply the same procedure to (1.12) after reducing it to its canonical form. Suppose, that in the hyperbolic case we have

$$\frac{\nu^3\mu^2}{\nu'^2+\mu^2}\neq 0$$

along the segment MN lying on the axis $r = 0$. Then a unique, twice continuously differentiable in r and φ solution of the stated problem exists in the region bounded by the characteristics of two families

$$2\sqrt{(\gamma+1)r} + \int \frac{\nu'^2+\mu^2}{\nu^3\mu^2}\,d\varphi = C_1; \quad 2\sqrt{(\gamma+1)r} - \int \frac{\nu'^2+\mu^2}{\nu^3\mu^2}\,d\varphi = C_1 \quad (1.14)$$

passing through M and N, provided that $\lambda(\varphi)$ has four continuous derivatives. In the elliptic case the statement of the problem is incorrect in the classical sense, and various regularizing methods [1] can be employed to obtain its solution.

When Ψ and Γ depend only on r (ν and μ are constant and the flow is steady), Fourier's method can be used to solve (1.12). The following approximate expression can be derived for the general case, at small r:

$$X \approx r\lambda + \frac{\gamma+1}{2}\frac{\nu^3\mu^2}{\nu'^2+\mu^2}(\lambda''+\lambda)\,r^2 \qquad (1.15)$$

(Expression for $\chi_{rr}(0,\varphi)$ is obtained analogously to those for $\Psi_{rr}(0,\varphi)$, and $\Gamma_{rr}(0,\varphi)$). Having found X in the small neighborhood of $r = 0$. $\Delta r = h \ll 1$, we can solve (1.12) in the hyperbolic region using the method of characteristics.

Thus the class of three-dimensional double waves allows the construction of some solutions of gasdynamic equations in the vicinity of a weak shock, provided that this shock is, for any t, a developable surface.

2. In the following, an essential part will be played by the transport equation of the discontinuities of the directional derivatives of the functions u_i and c, viz. $[u_{i\Phi}]$, and $[c_\Phi]$ along any bicharacteristic lying on the characteristic surface

$$\Phi(x_1,\ x_2,\ x_3) = t \qquad (2.1)$$

In the plane case when the surface $\Phi(x_1, x_2) = t$ is a surface of a weak shock propagating through an unperturbed gas with the velocity of sound, the transport equation is derived and investigated in [1] ([9] indicates the possibility of obtaining such an equation; in the case of a two-dimensional flow the transport equation for the discontinuities of derivatives along the characteristics was studied in detail in [10]).

In the following we shall assume that the characteristic surface (2.1) propagates with a normal velocity D uniform with respect to a fixed coordinate system (velocity of propagation relative to the gas is equal to the velocity of sound c; if on (2.1) $|\mathbf{u}| = 0$, i.e. a weak shock moves through the region of rest, we have $D = c$). Moreover we shall assume that the velocity of sound, and hence the density, are constant on the characteristic surface, and the velocity vector $\mathbf{u}$ is always orthogonal to the surface and its modulus is constant. These assumptions allow us to include in our study the motions behind the normal detonation waves when the Chapman – Jouguet condition

$$|\mathbf{u}| + c = D \qquad (2.2)$$

holds at the wave front, which means that the detonation wave moves with the velocity of sound relative to its combustion products. The function Φ satisfies the usual characteristic Eq.

$$(\Phi_1 u_1 + \Phi_2 u_2 + \Phi_3 u_3 - 1)^2 - c^2(\Phi_1^2 + \Phi_2^2 + \Phi_3^2) = 0 \qquad (2.3)$$

Using (2.2) together with the previously made assumptions, we can write (2.3) as

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = \frac{1}{D^2}, \qquad \Phi_i = \frac{\partial\Phi}{\partial x_i} \qquad (2.4)$$

Equations of the bicharacteristics can be written as

$$x_i^{\cdot} = c^2\Phi_i - u_i(u_1\Phi_1 + u_2\Phi_2 + u_3\Phi_3 - 1)$$
$$t^{\cdot} = -(u_1\Phi_1 + u_2\Phi_2 + u_3\Phi_3 - 1) \qquad (i = 1, 2, 3) \tag{2.5}$$

The right null vector of the characteristic matrix of the gas dynamic equations written for u_i and c, has the form

$$\mathbf{r} = (\Phi_1,\ \Phi_2,\ \Phi_3,\ \gamma - 1/2\sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2})$$

therefore we have the following relation for the discontinuities of the direction derivatives of the functions u_i and c ([9])

$$([u_{1\Phi}],\ [u_{2\Phi}],\ [u_{3\Phi}],\ [c_\Phi]) = \sigma\mathbf{r} \tag{2.6}$$

Here σ is some scalar function, and $[f_\Phi]$ corresponds, in the case of a weak shock, to the difference of two directional derivatives of the function f, taken at both sides of the shock (when one side is at rest, we have $[f_\Phi] = f_\Phi$ in the perturbed flow).

In the case of a normal detonation wave we have two distinct possibilities. First we shall consider the case when the derivatives of u_i and c are finite at the wave front (this is a case of secondary importance, since such flows can only occur behind the plane detonation waves). Let us put $[f_\Phi] = f_{1\Phi} - f_{2\Phi}$, where $f_{1\Phi}$ and $f_{2\Phi}$ are direction derivatives corresponding to two arbitrary flows behind the detonation wave of the given form. Jumps in $[f_\Phi]$ satisfy (2.6) when f denotes a basic function, and when the derivatives are finite, the case of normal detonation can be dealt with similarly to the case of a weak shock.

Let us now introduce new independent variables into the gasdynamic equations

$$\xi_i = x_i \quad (i = 1,\ 2,\ 3), \qquad \xi_4 = \Phi\ (x_1,\ x_2,\ x_3) - t \tag{2.7}$$

Utilising now the fact that the differentiation with respect to ξ_1, ξ_2 and ξ_3 will be directed inwards on the characteristic surface, we can carry out operations analogous to those performed in [1] (par. 3); equation defining σ can be written in its final form as

$$\sigma' + \frac{\gamma + 1}{2D^2}\sigma^2 + D\sigma(\Phi_{11} + \Phi_{22} + \Phi_{33})\left(\frac{c}{2} + \frac{\gamma - 1}{4}|\mathbf{u}|\right) = 0 \tag{2.8}$$

Here we have used t as a parameter on the bicharacteristic (σ' denotes a derivative of σ with respect to t), c = const is the velocity of sound and $|\mathbf{u}|$ = const is the modulus of the velocity vector on the surface $\Phi = t$. Eq. (2.8) is the Bernoulli's equation and it can always be integrated in quadratures, provided that an explicit expression for $\Delta\Phi$ as a function of t is known along the given bicharacteristic.

In the case of a weak shock ($|\mathbf{u}| = 0$, $c = D$) Eq. (2.8) becomes

$$\sigma' + (\gamma + 1)/(2c^2)\ \sigma^2 + c^2/2\ \sigma\ \Delta\Phi = 0 \tag{2.9}$$

Let us consider the case when the normal derivatives of u_i and c become infinite on the detonation wave front. We shall follow [8] (chapt. 6) and represent the first approximations of u_i and c as

$$u_i = S\ (\varphi)g_i\ (x_1,\ x_2,\ x_3,\ t) + q_i\ (x_1,\ x_2,\ x_3,\ t)$$
$$c = S\ (\varphi)\ g\ (x_1,\ x_2,\ x_3,\ t) + q\ (x_1,\ x_2,\ x_3,\ t)$$

Here $\varphi\ (x_1,\ x_2,\ x_3,\ t) = 0$ is the equation of the characteristic surface, $S\ (\varphi)$ is a certain generalized function such that $S\ (0) = 0$ and the derivative $S_{-1}\ (\varphi)$ of which becomes infinite when $\varphi = 0$, the functions g_i and g do not vanish and are sufficiently smooth on the characteristic surface, while functions q_i and q may include weaker singularities and their limiting values on $\varphi = 0$ coincide with the values of u_i and c on the wave front. We shall investigate the behavior of the functions g_i and g along the bicharacteristics on the surface $\varphi = 0$. It is obvious that outside the surface $\varphi = 0$ $(\Phi = t)$ functions g_i, g, q_i, and q are not uniquely definable.

Assuming that the expressions for u_i and c hold in some neighborhood of $\varphi = 0$ and equating to zero the coefficients of $S_{-1}(\varphi)$ in the system of gasdynamic equations (as it was done in [8] for a linear system), we obtain

$$(g_1, g_2, g_3, g) = \sigma_D \mathbf{r} \tag{2.10}$$

where $\sigma_D(x_1, x_2, x_3, t)$ is a scalar function.

Let us now assume that two distinct flows behind a detonation wave of the given form, are specified by g_i^*, g^* and g_i, g with the corresponding σ_{D^*} and σ_D respectively. Then, multiplying in the usual manner all equations by the components of the left null vector of the characteristic matrix, performing the summation and taking, subsequently, the difference of two relations obtained for two solutions with σ_D^* and σ_D, we obtain the following transport equation for $\sigma_R = \sigma_D^* - \sigma_D$ along the bicharacteristic

$$\sigma_R' + cD\frac{\gamma-1}{\gamma+1}\Delta\Phi\sigma_R = 0 \tag{2.11}$$

Let us now give a geometrical interpretation of $\Delta\Phi$. We know that, when a surface is given by Eq. $x_3 = F(x_1, x_2)$, then the radii of curvature R_1 and R_2 of the principal normal cross sections are defined by

$$(rt - s^2)R^2 - h[2pqs - (1+p^2)t - (1+q^2)r]R + h^4 = 0$$
$$\left(r = \frac{\partial^2 F}{\partial x_1^2},\ t = \frac{\partial^2 F}{\partial x_2^2},\ s = \frac{\partial^2 F}{\partial x_1 \partial x_2},\ p = \frac{\partial F}{\partial x_1},\ q = \frac{\partial F}{\partial x_2},\qquad h = \sqrt{1+p^2+q^2}\right) \tag{2.12}$$

Obtaining r, t, s, p and q from (2.1) in which we find an implicit expression for x_3 as a function of x_1 and x_2 and using the relation

$$2(\Phi_1\Phi_2\Phi_{12} + \Phi_2\Phi_3\Phi_{23} + \Phi_1\Phi_3\Phi_{13}) + \Phi_1^2\Phi_{11} + \Phi_2^2\Phi_{22} + \Phi_3^2\Phi_{33} = 0$$

derived from (2.4), we obtain

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2pqs - (1+p^2)t - (1+q^2)r}{h^3} = D\Delta\Phi \tag{2.13}$$

Thus $\Delta\Phi = 2H/D$ where H is the mean curvature of the surface (2.1).

Let us find $\Delta\Phi = f(t)$ along the bicharacteristic for the case of a weak shock. Assuming without loss of generality that $c = 1$, we can write the Eqs. of bicharacteristics as

$$x_i^{\cdot} = \Phi_i, \qquad t^{\cdot} = 1 \tag{2.14}$$

Since $\Sigma\Phi_k\Phi_{ik} = 0$, Φ_i are constant along any fixed bicharacteristic. Consequently the bicharacteristics are straight lines in the (x_1, x_2, x_3, t)-space. Utilizing the constancy of Φ_i and differentiating $\Delta\Phi$ along the bicharacteristic we obtain, with the help of (2.1), the expression

$$-(\Delta\Phi^{\cdot}) = (\Delta\Phi)^2 - \frac{2}{\Phi_1\Phi_2\Phi_3}(\Phi_1\Phi_{12}\Phi_{13} + \Phi_2\Phi_{21}\Phi_{23} + \Phi_3\Phi_{31}\Phi_{23}) \tag{2.15}$$

Next, using (2.1) we can obtain the following expression for the Gaussian curvature $K = 1/R_1R_2 = rt - s^2/h^4$:

$$K = \frac{1}{\Phi_1\Phi_2\Phi_3}(\Phi_1\Phi_{12}\Phi_{13} + \Phi_2\Phi_{21}\Phi_{23} + \Phi_3\Phi_{31}\Phi_{23}) \tag{2.16}$$

Further, the identities obtained by differentiating (2.1) twice with respect to all x_i and x_k and differentiation of K along the bicharacteristic (2.14), together yield

$$(K^{\cdot}) = -\Delta\Phi\, K \tag{2.17}$$

Finally, (2.15) and (2.17) together with (2.16) yield the following system of ordinary differential equations along a fixed bicharacteristic for $R_1(t)$ and $R_2(t)$:

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{\cdot}=-\frac{1}{R_1^2}-\frac{1}{R_2^2}, \qquad \left(\frac{1}{R_1R_2}\right)^{\cdot}=-\frac{1}{R_1R_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad (2.18)$$

General solution of (2.18) can be written in the form

$$R_1 = t + C_1, \qquad R_2 = t + C_2 \quad (2.19)$$

where C_i denote arbitrary constants. Inserting into (2.9)

$$\Delta\Phi = \frac{1}{t+C_1} + \frac{1}{t+C_2}$$

and integrating the result, we obtain the solution in the general form

$$\sigma = \frac{1}{\sqrt{t+C_1}\sqrt{t+C_2}\,[(\gamma+1)\ln(\sqrt{t+C_1}+\sqrt{t+C_2})+C]} \quad (2.20)$$

where C is an arbitrary constant. Formula (2.20) defines the principle of decay of partial derivatives of the solution with time, on the surface of a weak shock moving through the region of rest, in the three-dimensional case.

Using the Chapman-Jouguet condition, we can reduce the equations of bicharacteristics (2.5) in the case of normal detonation, to

$$\frac{dx_i}{d\lambda} = cD\Phi_i, \qquad \frac{dt}{d\lambda} = \frac{c}{D} \qquad (i = 1, 2, 3) \quad (2.21)$$

Inserting into it

$$\lambda = \frac{D}{c}\mu, \qquad x_i = Dx_i^*$$

we obtain, in place of (2.21) and (2.4),

$$\frac{dx_i^*}{d\mu} = \Phi_{x_i^*}, \qquad \frac{dt}{d\mu} = 1, \qquad \Phi^2_{x_1^*} + \Phi^2_{x_2^*} + \Phi^2_{x_3^*} = 1$$

respectively, and

$$\sum_{i=1}^{3}\Phi_{x_i^*x_i^*} = \frac{1}{t+C_1} + \frac{1}{t+C_2}, \qquad \Delta\Phi = \frac{1}{D^2}\left(\frac{1}{t+C_1}+\frac{1}{t+C_2}\right) \quad (2.22)$$

Inserting the expression obtained for $\Delta\Phi$ into (2.11) and integrating, we obtain

$$\sigma_R = C\,[(t+C_1)(t+C_2)]^{-\frac{c}{D}\frac{\gamma-1}{\gamma+1}} \quad (2.23)$$

We shall show in the next paragraph, that integration of (2.8) is pointless when $\Delta\Phi \neq 0$, while when $\Delta\Phi \equiv 0$, then the solution (2.8) has the form

$$\sigma_D = D^2\left(C + \frac{\gamma+1}{2}t\right)^{-1} \quad (2.24)$$

3. Let us now investigate the behavior of the partial derivatives of the basic functions on the characteristic surface, when the flow behind a weak shock or a normal detonation wave belongs to the class of double waves. We shall first consider the case, when the surface of a weak shock moves through the region of rest. The right null vector r of the characteristic matrix can be written, with the help of (1.5), as

$$\mathbf{r} = \left(\frac{\cos\varphi}{\nu}, \frac{\sin\varphi}{\nu}, \frac{\mu}{\nu}, \frac{\gamma-1}{2}\right) \tag{3.1}$$

while the vector representing the discontinuities of the directional derivatives will be

$$([u_{1\Phi}], [u_{2\Phi}], [u_{3\Phi}], [c_\Phi]) = -\left(\left[\frac{\partial u_1}{\partial t}\right], \left[\frac{\partial u_2}{\partial t}\right], \left[\frac{\partial u_3}{\partial t}\right], \left[\frac{\partial c}{\partial t}\right]\right)$$

Next we find $\partial u_1/\partial t$ and $\partial u_2/\partial t$ from (1.3) (by differentiation with respect to t and change to the polar coordinates) for $r = 0$, and use (3.1) and (2.6) to obtain the following expression for the scalar σ_d corresponding to the double wave type flow:

$$\sigma_d = \frac{1}{(\gamma+1)(t + B_d(\varphi))} \tag{3.2}$$

It should be noted that $B_d(\varphi) = \text{const}$ along a fixed bicharacteristic. Since the surface of a weak shock is, in this case, a developable surface, it follows that one of the radii of curvature of the principal normal cross sections becomes infinite along any bicharacteristic. Expression (2.20) is then replaced by

$$\sigma = \frac{1}{(\gamma+1)(t+B)} + \frac{1}{A(t+B)^{3/2} - (\gamma+1)(t+B)} \qquad (A,\ B = \text{const}) \tag{3.3}$$

For large t we obviously have

$$|\sigma - \sigma_d| = O\,(t^{-3/2}) \tag{3.4}$$

Let us consider, in the space of (x_1, x_2, x_3, t), the neighborhood Δ_k of a weak shock $\Phi(x_1, x_2, x_3) = t$, characterised by the fact that the distance ρ of any point M of this neighborhood along the surface $\Phi = t$, will be less or equal to k, i.e. $\rho(M, \Phi) \leqslant k$. Let us now assume that the perturbations present in the flow behind the weak shock, lag behind this shock (i.e. the flow is sufficiently smooth near the shock) and let the shock be also sufficiently smooth. Then, for $t \sim O(k^{-2/3})$ the difference between the similar first order partial derivatives appearing in any two solutions corresponding to the given form of the shock, will be of the order $O(k)$ (see (3.4)). Then, since the limiting values of u_i and c are identical for all the flows on $\Phi = t$, we find, from the Taylor expansions, that the basic gasdynamic quantities in Δ_k coincide with an accuracy of $O(k^2)$, when $t \sim O(k^{-2/3})$.

Thus, any flow behind a weak shock can be approximated by a double wave flow, provided that the time interval is sufficiently large.

Note. Using (2.20) we can construct any spherical flow near a weak shock travelling through a region of rest which will possess a self-similar flow of the triple wave type [3], and obtain analogous solutions.

Let us consider the case of a normal detonation. The boundary value problems for the system (1.1) when the flow behind a normal detonation wave belongs to the class of three-dimensional double waves, were stated in [4]. It was shown, that, in order to construct the flows, we must solve (1.1) with the initial conditions on the line $u_2 = f(u_1)$, the latter being a line on which the system is parabolic. However, this line is not a characteristic, and the system (1.1) is hyperbolic in the neighborhood of $u_2 = f(u_1)$. This problem is, in general, correct, and a unique solution can usually be found in the class of double waves, corresponding to the motion of a normal detonation wave represented by a developable surface at any value of t.

Initial conditions on the line $u_2 = f(u_1)$ have the form [4]

$$u_1^2+u_2^2+\Psi^2=A^2=\text{const},\qquad \Psi=u_1\Psi_1+u_2\Psi_2 \quad \text{for } \Psi$$
$$\Gamma=c^2/(\gamma-1)=\text{const},\qquad \Gamma_1u_1+\Gamma_2u_2=-Ac \quad \text{for } \Gamma \qquad (3.5)$$
$$X=0,\qquad X_2=F(X_2) \quad \text{for } \chi$$

where c is the velocity of sound, the functions f and F are arbitrary and are used to define the form of the wave at any instant of time. Using (3.5) and

$$u_1\Psi_{1i}+u_2\Psi_{2i}=u_1\Lambda_{1i}+u_2\Lambda_{2i}=0 \qquad (i=1,2)$$
$$(\Lambda=\Gamma+{}^1/_2(u_1^2+u_2^2+\Psi^2))$$

which follow from (3.5) and (1.1) we find, from (1.3), that all partial derivatives of the functions u_1 and u_2, and hence of u_3 and c, become infinite at the wave front if the wave is not of a plane form.

If we consider the complete gasdynamic equations and assume that

$$u_i=AD\Phi_i,\qquad c=\text{const}$$

are given on the surface $\Phi(x_1,x_2,x_3)=t$ of the detonation wave which is also a characteristic surface, we find that we can calculate all the inward derivatives of u_i and c on the surface $\Phi=t$ and obtain a non-homogeneous system of four linear equations defining the directional derivatives $u_{i\Phi}$ and c_Φ.

The determinant of the coefficients of $u_{i\Phi}$ and c_Φ vanishes, while the rank of the matrix of the expanded system, as shown by a direct check, is equal to four, provided that the detonation wave is not a plane wave. Thus we see, that the fact that the partial derivatives of the basic functions become infinite at the wave front, is the most important one in the investigation of the propagation of a normal detonation wave.

We shall use the transport equation (2.11) to correlate the general case of the flow behind a detonation wave with the double wave flow, assuming that g_i^+ and g^+ refer, in the expressions for u_i and c, to the double wave type flow, while g_i^- and g^- refer to the general type flow (provided that a flow different from the double wave type exists). Since (2.23)

$$\sigma_R=\sigma_D^+-\sigma_D^-=O\left(t^{-2\frac{c}{D}\frac{\gamma-1}{\gamma+1}}\right)$$

it is clear that the functions g_i^+, g^+ and g_i^-, g^- tend to the common limit with increasing t.

Thus we can also state in the case of a normal detonation, that an arbitrary flow near the detonation wave will, at large t, approximate a double wave in the sense that the principle describing how the derivatives of the basic functions decrease with increasing distance from the wave front, coincides with the corresponding law describing this phenomenon for a double wave.

If the detonation wave is a plane wave, it follows from (2.24) that a flow near such a wave will, at large t, approximate the self-similar Riemannian wave, since the following relation holds

$$\frac{\partial u}{\partial t}=-\frac{2D}{(\gamma+1)t}$$

Note. **Solution (2.23) for σ_k can be used to correlate arbitrary flows behind an expanding, normal spherical detonation wave with the self-similar solution due to Zel'dovich [10] of the problem on the expansion of a detonation wave from a single point. In this case we can also assume, for large t, that an arbitrary flow tends to a self-similar flow in the above sense.**

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